2019 Win Mid 1 v1

1. Consider the function:

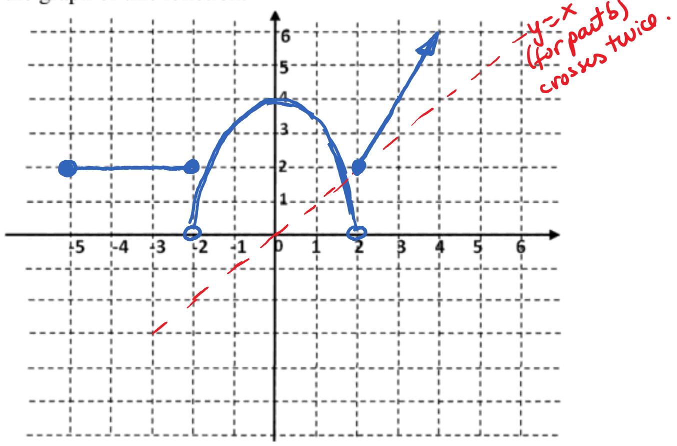
$$f(x) = \begin{cases} 2, & \text{if } -5 \le x \le -2 \\ -x^2+4, & \text{if } -2 < x < 2 \\ 2x-2, & \text{if } 2 \le x \end{cases}$$

horizontal line segment

concave down parabola, vertex at $(0,4)$, x-intercepts $(\pm 2, 0)$

line of slope 2, thru $(2,2)$

(a) (6 points) Sketch the graph of this function.



(b) (6 points) Find all solutions for the equation $f(x) = x$.

1) For $-5 \le x \le -2$: no solution ($x=2$ is not in the domain)

2) For $-2 < x < 2$: $-x^2+4 = x$

$x^2 + x - 4 = 0$

Q.F.: $x = \dfrac{-1 \pm \sqrt{17}}{2}$

$\dfrac{-1-\sqrt{17}}{2} \cong -2.56$ not in $(-2,2)$

$\dfrac{-1+\sqrt{17}}{2} \cong 1.56$ ✓

3) For $x \geq 2$: $2x - 2 = x$

$x = 2 \geq 2$ ✓

2 solutions:

ANSWER: $x =$ $\boxed{\dfrac{-1+\sqrt{17}}{2} \cong 1.56, \quad 2}$

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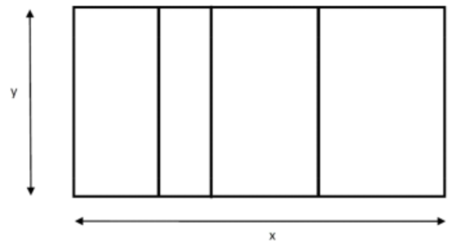
2. (12 points) Sammy has a total length of 400 meters of fence and wants to use this fence to build a rectangular enclosure with 4 compartments, as shown in the picture below.

   What dimensions x and y should Sammy use for the enclosure in order to maximize the total area inside the enclosure, and what is the largest area that can be enclosed? Include correct units.



fence = 400 m = $2x + 5y$ => $y = 80 - \frac{2}{5}x$
$= 80 - 0.4x$

area $A(x) = xy = x(80 - 0.4x)$
$= -0.4x^2 + 80x$

This is a quadratic function with $a < 0$ so it will be maximal at its vertex:

$$x = -\frac{b}{2a} = \frac{-80}{2(-0.4)} = 100$$

Other dimension:

$$y = 80 - 0.4x = 80 - 40 = 40$$

Max area = $x \cdot y = (100)(40) = 4000$

OR:

$x = 200 - 2.5y$

$A(y) = -2.5y^2 + 200y$

$y = -\frac{b}{2a} = \frac{-200}{2(-2.5)}$
$= \frac{200}{5} = 40.$

$x = 200 - 2.5(40)$
$= 100$

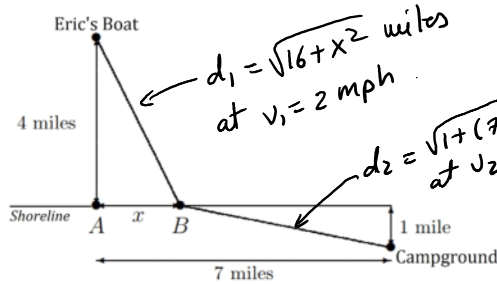
max area = $(100)(40)$
$= 4000.$

ANSWER (include units): maximal area = 4000 m^2

achieved when $x =$ 100 m and $y =$ 40 m

3. (12 points) Eric is in a rowboat located 4 miles north of point A on the shore of a lake. He wants to go to a campground located 7 miles east and 1 mile south of the point A. To get there, Eric first paddles at 2 miles per hour in a straight line to a point B, located on the shore x miles east of point A, and then walks in a straight line from B to the campground, at 4.5 miles per hour. His path is pictured below.

   (a) (6 points) Find a function $T(x)$ that gives the total time it takes Eric to get to the campground as an expression of x.



$d_1 = \sqrt{16+x^2}$ miles at $v_1 = 2$ mph

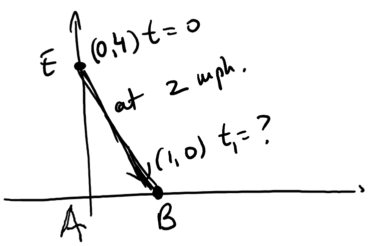
$d_2 = \sqrt{1+(7-x)^2}$ at $v_2 = 4.5$ mph

$$\text{time} = t_1 + t_2 = \frac{d_1}{v_1} + \frac{d_2}{v_2}$$

Using coordinates or the Pythagorean Thm: $d_1 = \sqrt{16+x^2}$, $d_2 = \sqrt{1^2+(7-x)^2}$

ANSWER: $T(x) = \boxed{\dfrac{\sqrt{16+x^2}}{2} + \dfrac{\sqrt{1+(7-x)^2}}{4.5}}$

   (b) (6 points) Take point A as the origin of a coordinate system and assume Eric reaches the shore at point B that's $x = 1$ mile away from A. Write parametric equations for Eric's coordinates as functions of time t for the portion of his trip when he's paddling towards point B.



$EB = \sqrt{16+1} = \sqrt{17}$

$t_1 = \dfrac{\sqrt{17}\text{ miles}}{2\text{ mph}} = \dfrac{\sqrt{17}}{2}\text{ hrs} \approx 2.0616$ hrs.

x coord: $v_x = \dfrac{\Delta x}{\Delta t} = \dfrac{1}{(\sqrt{17}/2)} = \dfrac{2}{\sqrt{17}} \approx 0.4851$ mph.

$x_0 = 0$

y-coord: $v_y = \dfrac{\Delta y}{\Delta t} = \dfrac{-4}{(\sqrt{17}/2)} = \dfrac{-8}{\sqrt{17}} \approx -1.9403$ mph.

ANSWER: $x(t) = \boxed{(2/\sqrt{17})t \approx 0.49t}$

$y(t) = \boxed{4 - (8/\sqrt{17})t \approx 4 - 1.94t}$

4. (14 points) Lassie is a dog and she is looking for her owner, a boy named Timmy. Timmy has fallen

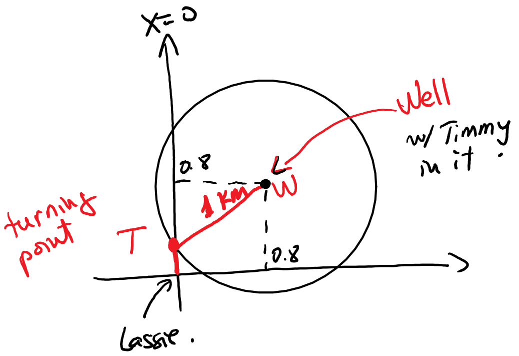
4. (14 points) Lassie is a dog and she is looking for her owner, a boy named Timmy. Timmy has fallen in a well and he is blowing a whistle that can be heard from up to 1 km away. Initially, Lassie is at a point 0.8 km south and 0.8 km west of the well.

   (a) (4 points) Can Lassie hear Timmy's whistle from her initial point? Justify your answer.

No. The distance from Lassie to Timmy is:

$$\sqrt{(0.8)^2+(0.8)^2} = \sqrt{1.28} \cong 1.1314 \text{ km} > 1 \text{ km}$$

   (b) (10 points) Lassie starts running due north, at a constant speed of 20 km/hr. After a while, she hears the whistle and as soon as she hears it she turns and continues running at the same speed in a straight line towards the well. How long does it take Lassie to reach Timmy, in minutes?



w/ origin at Lassie's position:

$$\begin{cases} \text{whistle range is: } (x-0.8)^2+(y-0.8)^2=1 \\ \text{Lassie's north path is: } x=0 \end{cases}$$

Intersecting: $0.64+(y-0.8)^2=1$

$(y-0.8)^2=0.36$

$y-0.8=\pm\sqrt{0.36}=\pm 0.6$

$\boxed{y=0.2}$ or 1.4

y-coordinate of turning point.

Lassie travels: $LT+TW$

$= 0.2 + 1 = 1.2$ km, at 20 km/hr.

$\therefore t = \dfrac{1.2 \text{ km}}{20 \text{ km/hr}} = 0.06 \text{ hrs} = 3.6$ minutes

(we could compute $TW = 1$ km via the distance formula, but it's quicker to just notice that it's the radius of the whistle sound)

ANSWER: $t =$ 3.6 minutes